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D-instanton probes of $\mathcal{N} = 2$ non-conformal geometries

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Abstract: D-instanton calculus has proved to be able to probe the AdS near horizon geometry for N D-branes systems which, when decoupled from gravity, yield four-dimensional superconformal gauge theories with various matter content. In this work we extend previous analysis to encompass fractional brane models which give rise to non-conformal $\mathcal{N} = 2$ Super Yang-Mills theories. Via D-instanton calculus we study the geometry of such models for finite N and recover the β function of the gauge coupling constants which is expected in non-conformal gauge theories. We also give a topological matrix theory formulation for the D-instanton action of these theories. Finally, we revisit the related system where the D3-branes wrap a $\mathbb{R}^4/\mathbb{Z}_p$ orbifold singularity and the D(-1)-branes are associated to instanton solutions of four-dimensional gauge theories in the blown down ALE space.

Keywords: D-branes, Brane Dynamics in Gauge Theories, AdS/CFT Correspondance.

Contents

1. Introduction

The present work arises from two distinct sets of motivations which point both in the direction of D-branes.

The first motivation is given by the difficulties that one finds in extending the Maldacena's conjecture [1] to $\mathcal{N}=2$ models which do not enjoy the property of conformal invariance [2]–[5]. The classical supergravity solutions dual to such non-conformal gauge theories typically possess infrared singularities which have to be resolved to yield sensible models. A proposal for a resolution of such singularities is known as the *enhançon* mechanism: if a massive probe moves in the backgrounds of the fractional D3-brane, it becomes tensionless before reaching the singularity. The supergravity description breaks down and one is forced to consider stringy effects which presumably would smooth out the singularity. Despite some recent progresses [6] in the understanding of the physics of the enhançon, a complete satisfactory picture is still missing.

In the conformal versions of the Maldacena's conjectures D-instanton calculus has been proved to efficiently probe the geometry of the associated supergravity solution [7]. Indeed instanton dominated correlators for the four-dimensional Yang-Mills theory (SYM_4) with color gauge group $\text{SU}(N)$ and sixteen supersymmetric charges have been put in correspondence in the $N \to \infty$ limit with D-instanton corrections to higher derivative terms in type-IIB supergravity on $AdS_5 \times S^5$ [7, 8]. The result relies on the fact, proved in [7], that the multi-instanton moduli space of $\mathcal{N} = 4$ SU(N) SYM$_4$ factors out, in the large-N limit, in an $AdS_5 \times S^5$ term describing the overall center of mass degrees of freedom and an effective SU(k) matrix model describing degeneracies of the bound state of k D-instantons in ten dimensions. Generalizations to other SYM$_4$ with different gauge group or matter content have been addressed in [9, 10, 11]. In [9] the authors studied a model arising from a system of N D3-branes living at an orientifold 7-plane together with eight D7-branes [12] that, in the low energy limit and when gravity is decoupled, gives rise to superconformal $\mathcal{N} = 2$ SYM$_4$ with gauge group Sp(N). Depending on whether one tests the geometry with "regular" or "fractional"[1] D(-1)-instantons, the expected large-N geometry $AdS_5 \times S^5/\mathbb{Z}_2$ or $AdS_5 \times S^3$ were recovered, respectively. The case of IIB string theory compactified to six dimensions on K3 with a vanishing two cycle was coped with in [10]. The low energy theory is $\mathcal{N} = 2$ SYM$_4$ with gauge group SU(N) and $N_F = 2N$ fundamental hypermultiplets. The geometry seen by the instanton probe was proved to be $AdS_5 \times S^1$ in the large-N limit.

A more general class of models is defined by locating a stack of D3-branes at a $\mathbb{R}^6/\Gamma$ singularity, with Γ a discrete subgroup of SU(4). From the gauge theory point of view such moddings act on the $\mathcal{R}$-symmetry group and with a suitable prescription they reduce the number of supersymmetric charges without spoiling the conformal invariance of the theory [13]. The resulting low energy theory is $\mathcal{N} = 0, 1, 2$ SYM$_4$ (depending on whether Γ is embedded in a SU(4), SU(3) or SU(2) subgroup of the SU(4)$_\mathcal{R}$ $\mathcal{R}$-symmetry group, respectively), with product gauge groups and bifundamental matter. From the point of view of instanton calculus, such theories, for $\Gamma = \mathbb{Z}_p$, were studied in [11] where the near horizon geometry, $AdS_5 \times S^5/\mathbb{Z}_p$, was recovered in the large-N limit.

One of the aims of this paper will be the study, via a *finite* N D-instanton calculus, of a non-conformal variant of the models we just described, involving both regular and fractional branes. We focus ourselves on the $\mathcal{N} = 2$ cases but most of our analysis extends to more general situations. The deviations from conformal invariance seen from the probe point of view will be shown to be entirely determined by the one-loop β-function coefficients of the underlying gauge theory. The analysis of non-trivial RG-flows in the context of non-conformal $\mathcal{N} = 2$ AdS/QFT correspondences of the kind we study was pioneered in [3]. The subsequent results [5, 14, 15] strongly motivate the present work. In [14] the correspondence between the one-loop beta function coefficient and the tadpole of the dilaton in the bulk supergravity theory was tested in an $\mathcal{N} = 0$ SYM$_4$ gauge theory constructed out of D-branes and O-planes.

[1] By a "fractional" brane in the unoriented context we mean an unpaired D-brane constrained to live on the orientifold plane. A "regular" brane can move off the fixed point of the orientifold if it does not break the modding discrete symmetry.

The result was extended in [15] to various $\mathcal{N} = 0, 1, 2$ non-conformal geometries. In this reference the one-loop beta function coefficient is read from the annulus/cylinder string computation which is shown to be protected from contributions coming from massive string modes, admitting therefore equivalent description in terms of either supergravity or SYM_4 degrees of freedom. In [5] the authors constructed the explicit supergravity solution of a fractional D3-brane outside of the enhançon and showed how the expected running of the gauge coupling constant is recovered from it. In this work we study the $\mathcal{N} = 2$ RG-flow from the D-instanton probe perspective.

The second motivation for this paper was induced by the desire to find a physicist's handle to some beautiful mathematical properties enjoyed by the moduli space of gauge connections realized *à la* ADHM [16]. Our present poor understanding of such properties accounts, in our opinion, for the difficulties we have in computing non-perturbative effects in SYM_4 theories. This task turns out to be one of the hardest challenges of present days theoretical physics. In spite of some spectacular progresses [17], the only tool to study non-perturbative properties in the framework of field theory is via instanton calculus. The single hurdle that makes such computations so difficult is given by the presence of constraints in the ADHM construction of the moduli space of gauge connections. Such constraints can be explicitly solved only in a very limited number of instances but even in those cases the amount of algebra involved in the computations tends to obscure its physical meaning. To overcome this difficulty it has been proposed [18] to implement such constraints in the functional integration via the introduction of a suitable number of Dirac deltas. Given that this prescription reproduces the correct measure (and it does [19]), we can now write the instanton measure for arbitrary winding number. This strategy has proved very successful especially in giving non-perturbative checks [7] of the Maldacena's conjecture [1]. In this last instance, the absence of a vacuum expectation value for the scalars of the theory and the large-N limit conspire to give the desired result. Such combination of favorable events is not always present. For example, giving a non-trivial v.e.v. to the scalars corresponds in the brane picture to separate the branes, and this gives rise to less symmetric configurations rendering the computations much more involved. This is why some of the authors have proposed to exploit the topological properties of SYM_4 [20], an approach that we extend here also to the various $\mathcal{N} = 2$ models arising from the above mentioned brane configurations; an extension to $\mathcal{N} = 4$ has been carried out in [21]. The localization properties which show up in the topological formulation give indeed useful simplifications. One of the most striking outcome in the first checks [22] of the non-perturbative behaviour of $\mathcal{N} = 2$ SYM_4 computed by Seiberg and Witten was in fact its agreement with semi-classical instanton calculus. This in turn implies that the saddle point approximation for $\mathcal{N} = 2$ SYM_4 is exact: a typical feature of topological theories. Along this line it is possible to show that the relevant correlators can be written as a total derivative in the moduli space of gauge connections: only zero size instantons contribute to the computation. The measure is then recovered as a change of coordinates from fermionic to bosonic variables in the supersymmetric extension of the hyperkähler quotient construction [19]. But where does the property of being hyperkähler come from? Formally it can be obtained considering the infinite dimensional hyperkähler quotient of the space of general connections by the triholomorphic action of the group of gauge transformations $\mathcal{G}$. The zero level set is given by the space of self-dual solutions which, quotiented by $\mathcal{G}$, yields the moduli space. A much more palatable derivation can be given using D-branes. The ADHM construction for SYM_4 with sixteen supercharges, gauge group $\mathrm{SU}(N)$ and arbitrary instanton number k can indeed also be obtained from the low-energy effective action of a system formed by a stack of k Dp-branes and N D$(p+4)$-branes reduced to zero space-time dimensions when gravity is switched off [7, 23, 24]. In this set up the familiar hyperkähler properties of the ADHM instanton moduli space descends naturally from similar structures in the underlying D(-1)–D3-supersymmetric gauge theory (see [25] for a review from the field theory perspective). Such intimate relationship between the ADHM construction and D-branes lies also at the origin of the results which we commented before relating moduli spaces of gauge connections and near horizon AdS geometry.

This is the plan and summary of the paper: in section 2 we collect the main background material needed in the rest of the paper. Section 3 is devoted to the study of the geometry associated to various non-conformal $\mathcal{N}=2$ gauge theories living on D3-brane systems located at $\mathbb{R}^4/\mathbb{Z}_p$ singularities. As usual, the tested geometry depends on whether we use fractional or regular instanton probes. In the former case we found a geometry which is a sort of non-conformal deformation of $AdS_5 \times S^1$. The position in the AdS_5 space is identified with the energy scale of the boundary gauge theory and the β-function coefficients agree with the field theory expectations. In the case where we use regular instantons as a probe, the geometry turns out to be $AdS_5 \times S^5/\mathbb{Z}_p$ independently on whether the theory is conformal or not. This is in agreement with our expectations from both supergravity and field theory point of view, since, as we will see, regular instantons test only the overall sum of the coupling constants in the product gauge group, and it is precisely this combination that does not run. We stress the fact that all results are valid already at finite N! In section 4, we recast the D(-1)–D3 lagrangians as a topological $0+0$-dimensional theory of ADHM constraints and clarify the connection with the results of [20]. The fact that the multi-instanton measure can be related to the partition function of a topological gauge theory encourage us to believe that the cumbersome integrations over the multi-instanton moduli space could be performed directly along the lines of [26]. Finally in section 5, we consider the case where the D3-branes wrap the $\mathbb{R}^4/\mathbb{Z}_p$ singularity. D(-1) instantons are known [27] to realize the Kronheimer-Nakajima construction of self-dual solutions of SYM_4 living on a blown down ALE space. Following the line of our previous analysis we show how the expected $AdS_5/\mathbb{Z}_p \times S^5$ geometry, with $\mathbb{Z}_p$ acting on the AdS boundary, is recovered from the probe perspective. In the appendices we have collected some material concerning the spinorial representation of SO(10) which set up the notation of section 2, and a rederivation of the action of section 4 from orbifold projection on the $\mathcal{N}=4$ topological action of [21].

2. Instanton moduli spaces from D-branes dynamics

The starting point of our discussion are the results of [7] for the measure and multi-instanton action in the $\mathcal{N}=4$ case[2] which, when suitably modified, will lead to the models of interest in this paper. The basic objects in the ADHM construction [16] of SU(N) instanton solution in four dimensions are the $[N+2k]\times[2k]$ and $[2k]\times[N+2k]$ matrices

$$\Delta(x) = a + bx\,, \qquad \bar{\Delta}(x) = \bar{a} + \bar{x}\bar{b}\,, \tag{2.1}$$

where $x_{\alpha\dot{\alpha}} = x_m \sigma^m_{\alpha\dot{\alpha}}$, $\bar{x}^{\dot{\alpha}\alpha} = x^m \bar{\sigma}^{\dot{\alpha}\alpha}_m$,[3] is the position of the multi-instanton center of mass and all the remaining moduli are collected in the matrix a (see formula (2.3) below). Finally b is a $[N+2k]\times[2k]$ matrix which can be conveniently chosen to be

$$b = \begin{pmatrix} 0 \\ \mathbf{1}_{[2k]\times[2k]} \end{pmatrix}, \qquad \bar{b} = \begin{pmatrix} 0, & \mathbf{1}_{[2k]\times[2k]} \end{pmatrix}. \tag{2.2}$$

The moduli space of the solutions to the self-dual equations of motion is characterized in terms of the supercoordinates

$$a \equiv \begin{pmatrix} w_{u\,i\dot{\alpha}} \\ a_{\alpha\dot{\alpha}\ \ li} \end{pmatrix}, \qquad \mathcal{M}^A \equiv \begin{pmatrix} \mu^A_{iu} \\ \mathcal{M}^A_{\beta\ \ li} \end{pmatrix} \tag{2.3}$$

satisfying the bosonic and fermionic ADHM constraints

$$\begin{aligned} \bar{\Delta}\Delta &= f^{-1}_{k\times k}\mathbf{1}_{[2]\times[2]}\,, \\ \bar{\Delta}\mathcal{M}^A &= \bar{\mathcal{M}}^A\Delta\,, \end{aligned} \tag{2.4}$$

with $f_{k\times k}$ an invertible $[k]\times[k]$ matrix.

The solutions to the self-dual equations of motion for the various fields in the $\mathcal{N}=4$ vector multiplet are given by [7]

$$\begin{aligned} A_n &= \bar{\mathrm{U}}\ \partial_n\ \mathrm{U}\,, \\ \Psi^A &= \bar{\mathrm{U}}\left(\mathcal{M}^A f\bar{b} - bf\bar{\mathcal{M}}^A\right)\mathrm{U}\,, \\ i\,\Phi^{AB}(x) &= \frac{1}{2\sqrt{2}}\bar{U}\,(\mathcal{M}^B f\bar{\mathcal{M}}^A - \mathcal{M}^A f\bar{\mathcal{M}}^B)U + \\ &\quad + \bar{U}\cdot\begin{pmatrix} 0_{[N]\times[N]} & 0_{[N]\times[2k]} \\ 0_{[2k]\times[N]} & L^{-1}\Lambda^{AB}_{[k]\times[k]}\otimes 1_{[2]\times[2]} \end{pmatrix}\cdot U \end{aligned} \tag{2.5}$$

[2]We denote by $\mathcal{N}$ one quarter of the number of real supercharges.

[3]We adopt the following convention for the sigma matrices: $\sigma^m = (\sigma^c, -i\mathbf{1})$ and $\bar{\sigma}^m = (-\sigma^c, -i\mathbf{1})$.

in terms of the kernels $U_{[N+2k]\times[N]}$, $\bar{U}_{[N]\times[N+2k]}$ of the ADHM matrices $\bar{\Delta}, \Delta$. In (2.5), Λ^{AB} is the fermionic bilinear

$$\Lambda^{AB} = \frac{1}{2\sqrt{2}} \left(\bar{\mathcal{M}}^A \mathcal{M}^B - \bar{\mathcal{M}}^B \mathcal{M}^A \right) , \tag{2.6}$$

and the operator L is defined as

$$L \cdot \Omega = \frac{1}{2} \left\{ W^0, \Omega \right\} + [a_m, [a_m, \Omega]] , \tag{2.7}$$

with $W^0_{ij} = \bar{w}^{\dot{\alpha}}_{iu} w_{uj\dot{\alpha}}$. In the above equations the indices $i, j = 1, \ldots, k$, $u, v = 1, \ldots, N$, $\alpha, \dot{\alpha} = 1, 2$ and $A = 1, \ldots, 4$ label the instanton number, the color, SO(4) and $\mathrm{SO}(6)_{\mathcal{R}} \cong \mathrm{SU}(4)_{\mathcal{R}}$ quantum numbers, respectively.

The measure of the moduli space and the multi-instanton action can be efficiently read from the partition function of the gauge theory governing the low energy dynamics of a system of k D(-1) and N D3-branes moving in flat space [7]

$$\begin{aligned} Z_{k,N} = \frac{1}{\mathrm{Vol}\ \mathrm{U}(k)} \int & d^{6k^2}\chi \, d^{8k^2}\lambda \, d^{3k^2}D \, d^{4k^2}a \, d^{8k^2} \times \\ & \times \mathcal{M} \, d^{2kN}w \, d^{2kN}\bar{w} \, d^{4kN}\mu \, d^{4kN}\bar{\mu} \, e^{-S_{k,N}} , \end{aligned} \tag{2.8}$$

after integrating out the $(\chi_a, \lambda_{\dot{\alpha}A}, D_c)$ degrees of freedom associated to the U(k) vector multiplet, where $a = 1, \ldots, 6$ stands for the vector index in the transverse $\mathrm{SO}(6)_{\mathcal{R}}$ and $c = 1, 2, 3$. This gauge theory is defined by the dimensional reduction to 0+0 dimensions of the $\mathcal{N} = 1$ U(k) gauge theory in $D = 6$ with one hypermultiplet transforming in the adjoint representation, denoted by $(a_{\alpha\dot{\alpha}}, \mathcal{M}^A_{\alpha})$ and N transforming in the fundamental representation (and its conjugate), denoted by $(w_{\dot{\alpha}}, \mu^A; \bar{w}^{\dot{\alpha}}, \bar{\mu}^A)$. The action can be written as

$$S_{k,N} = \frac{1}{g_0^2} S_G + S_K + S_D \tag{2.9}$$

with

$$\begin{aligned} S_G &= \mathrm{tr}_k \left(-[\chi_a, \chi_b]^2 + \sqrt{2} i\pi \lambda_{\dot{\alpha}A} [\chi^{\dagger}_{AB}, \lambda^{\dot{\alpha}}_{B}] - D^c D^c \right) , \\ S_K &= -\mathrm{tr}_k \left([\chi_a, a_n]^2 - \chi_a \bar{w}^{\dot{\alpha}} w_{\dot{\alpha}} \chi_a + \sqrt{2} i\pi \mathcal{M}^{\alpha A} [\chi_{AB} \mathcal{M}^B_{\alpha}] - 2\sqrt{2} i\pi \chi_{AB} \bar{\mu}^A \mu^B \right) , \\ S_D &= \mathrm{tr}_k \left(i\pi \left(-[a_{\alpha\dot{\alpha}}, \mathcal{M}^{\alpha A}] + \bar{\mu}^A w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}} \mu^A \right) \lambda^{\dot{\alpha}}_A + \right. \\ & \qquad \left. + D^c \left(\bar{w} \tau^c w - i \bar{\eta}^c_{mn} [a_m, a_n] \right) \right) . \end{aligned} \tag{2.10}$$

Given the classical group isomorphism $\mathrm{SO}(6)_{\mathcal{R}} \cong \mathrm{SU}(4)_{\mathcal{R}}$, $\mathrm{SO}(6)_{\mathcal{R}}$ vectors can also be written as $\chi_{AB} \equiv \frac{1}{\sqrt{8}} \Sigma^a_{AB} \chi_a$ with the $\Sigma^a_{AB} = (\eta^c_{AB}, i\bar{\eta}^c_{AB})$, $\bar{\Sigma}^a_{AB} = (-\eta^c_{AB}, i\bar{\eta}^c_{AB})$ given in terms of the t'Hooft symbols. In the limit $g_0 \to \infty$, gravity decouples from the gauge theory and the contributions coming from S_G are suppressed; then the fields

$\lambda^A_{\dot\alpha}$, D_c become lagrangian multipliers implementing the ADHM constraints (2.4) in the form

$$\begin{aligned} \bar\mu^A w_{\dot\alpha} + \bar w_{\dot\alpha}\mu^A - [a_{\alpha\dot\alpha}, \mathcal{M}'^{\alpha A}] &= 0\,, \\ \bar w \tau^c w - i\bar\eta^c_{mn}[a_m, a_n] &= 0\,. \end{aligned} \tag{2.11}$$

An explicit evaluation of the remaining integrations for generic k seems to be at present out of reach. However, in the limit of large-N significant simplifications take place. This program has been carried out in [7]. The results show that the degrees of freedom associated to the k-instanton center of mass are described by a position in an $AdS_5 \times S^5$ space, while the dynamics of the excitations around this configuration are governed by a $0+0$ gauge theory obtained from the dimensional reduction of a $\mathcal{N}=1$ SU(k) gauge theory in $D=10$ to zero space-time dimensions.

In the following we apply these techniques to non-conformal gauge theories and show how to make contact with the geometry of the corresponding supergravity backgrounds already at finite N.

3. On the geometry of the non-conformal case

Having seen how the ADHM moduli space for $\mathcal{N}=4$ arises from a D(-1)-D3-brane system,[4] this section is devoted to the study of gauge theories living on D3-branes located at a $\mathbb{R}^4/\Gamma$ orbifold singularity, Γ being a discrete subgroup of SU(2).

3.1 Pure $\mathcal{N}=2$ Super Yang-Mills

We first discuss the case of pure $\mathcal{N}=2$ SU(N) SYM$_4$ to exemplify our strategy, postponing to the next subsection the most general case corresponding to gauge theories with product groups and bifundamental matter.

In the following our D3-branes will be longitudinal to the 7, 8, 9, 10 directions of space time and transverse to the remaining ones. In this section discrete symmetries act on the plane formed by the 1, 2, 4, 5 directions leaving 3, 6 unaltered. The low energy dynamics for a stack of N fractional D3-branes lying at a $\mathbb{R}^4/\Gamma$ singularity is governed by a pure $\mathcal{N}=2$ SU(N) SYM$_4$. This theory can be obtained by means of a Γ-projection of the previously discussed $\mathcal{N}=4$ SU(N) gauge theory associated to N nearby D3-branes moving on a flat space-time. In the $\mathcal{N}=2$ language, fields in the vector multiplet of the parent $\mathcal{N}=4$ theory are invariant under Γ, while the whole hypermultiplet, whose fields roughly describe the positions of the branes in the orbifold space, is projected out. Fractional D3-branes are then stuck at the origin of the orbifold space. Alternatively one can think of these branes as D5-branes wrapping a vanishing (at the orbifold point) two-cycle of a blown down ALE space.

[4] Our choice for the dimensionality of the branes is taken for the sake of clarity and will not spoil the generality of our discussion.

The aim of this section is to show how the moduli space measure and multi-instanton action of pure $\mathcal{N}=2$ SU(N) Super Yang-Mills theory can be read from the low energy lagrangian governing the dynamics of a fractional D(−1)–D3 system on $\mathbb{R}^4/\mathbb{Z}_p$. The low energy excitations of such a system are defined by the Γ-projection [27], which in the notation of [11] reads

$$\begin{aligned}
\mu^A &= e^{2\pi i \frac{q_A}{p}} \mu^A\,, & \bar\mu^A &= e^{2\pi i \frac{q_A}{p}} \bar\mu^A\,,\\
\mathcal{M}^A_\alpha &= e^{2\pi i \frac{q_A}{p}} \mathcal{M}^A_\alpha\,, & \lambda^A_{\dot\alpha} &= e^{2\pi i \frac{q_A}{p}} \lambda^A_{\dot\alpha}\,,\\
\chi^{AB} &= e^{2\pi i \frac{q_A+q_B}{p}} \chi^{AB} & &
\end{aligned} \tag{3.1}$$

with $q_1=q_2=0$, $q_3=-q_4=1$. All the other fields $(w_{\dot\alpha}, \bar w^{\dot\alpha}, a_m, D^c)$ remain invariant under the $\mathbb{Z}_p$ orbifold action.

The net effect of the projection (3.1) is to break the $\mathcal{R}$-symmetry group of the $\mathcal{N}=4$ theory down to down to $\mathrm{SU}(2)_{\dot A}\times \mathrm{U}(1)_{\mathcal{R}}$, where $\mathrm{SU}(2)_{\dot A}$ is the $\mathcal{N}=2$ automorphism group and $\mathrm{U}(1)_{\mathcal{R}}$ the anomalous $\mathcal{R}$ charge. A more detailed discussion of this point can be found in the appendix A.

After the Γ-projection (3.1) the multi-instanton action can be read from (2.10) with fermionic indices A,B now restricted to $\dot A, \dot B = 1,2$ (in the fundamental of the automorphism group) and the bosonic components $(\chi_1,\chi_2,\chi_4,\chi_5)$ set to zero. The remaining components will be denoted by the complex notation

$$\begin{aligned}
\phi &\equiv 2\chi_{43} = 2\chi^{12} = \frac{1}{\sqrt{2}}\left(-\chi_3 + i\chi_6\right),\\
\bar\phi &\equiv 2\chi_{21} = 2\chi^{34} = \frac{1}{\sqrt{2}}\left(-\chi_3 - i\chi_6\right).
\end{aligned} \tag{3.2}$$

We can now follow [7, 10] in order to determine a gauge invariant measure. We start by defining the SU(N) gauge invariant components $W^m_{ij}, \zeta^{\dot\alpha\dot A}_{ij}$ through

$$\begin{aligned}
W^0_{ij} &= \bar w^{\dot\alpha}_{iu}\, w_{uj\dot\alpha}\,, \qquad W^c_{ij} = \bar w^{\dot\alpha}_{iu}\, \tau^{c\dot\beta}{}_{\dot\alpha}\, w_{uj\dot\beta}\,, \qquad c=1,2,3\,,\\
\mu^{\dot A}_{iu} &= w_{uj\dot\alpha}\zeta^{\dot\alpha\dot A}_{ji} + \nu^{\dot A}_{iu}\,, \qquad \text{with } \bar w^{\dot\alpha}_{iu}\nu^{\dot A}_{uj} = 0\,,\\
\bar\mu^{\dot A}_{iu} &= \bar\zeta^{\dot A}_{\dot\alpha ij}\bar w^{\dot\alpha}_{uj} + \bar\nu^{\dot A}_{iu}\,, \qquad \text{with } \bar\nu^{\dot A}_{iu} w_{uj\dot\alpha} = 0
\end{aligned} \tag{3.3}$$

and perform the integrations over the iso-orientation modes (parameterizing a point in the coset space $\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-2k)\times \mathrm{U}(1)}$) and over their "fermionic superpartners" $\nu^{\dot A}$, $\bar\nu^{\dot A}$

$$\begin{aligned}
\int_{\substack{\text{gauge}\\ \text{coset}}} d^{2kN}w\, d^{2kN}\bar w &= c'_{k,N}\left(\det_{2k} W\right)^{N-2k} d^{k^2}W^0 d^{3k^2}W^c\,,\\
\int d^{2k(N-2k)}\nu d^{2k(N-2k)}\bar\nu \exp\left[\sqrt{8}\pi i \mathrm{tr}_k \chi_{\dot A\dot B}\bar\nu^{\dot A}\nu^{\dot B}\right] &= (-2\pi^2)^{k(N-2k)}\left(\det_k \bar\phi\right)^{2(N-2k)}
\end{aligned} \tag{3.4}$$

with

$$c'_{k,N} = \frac{2^{2kN-4k^2+k}\pi^{2kN-2k^2+k}}{\prod_{i=1}^{2k}(N-i)!}\,. \tag{3.5}$$

Furthermore, by integrating out the auxiliary field D^c we enforce the bosonic ADHM constraint $W^c = i\bar\eta^c_{mn}[a_n, a_m]$, while the integration on $\lambda^{\dot A}_{\dot\alpha}$ enforces the fermionic constraint and give rise to a factor π^{4k^2} in the partition function. Plugging (3.3) and (3.4) into the $\mathbb{Z}_p$ projected analog of (2.8) one is left with the SU(N) invariant measure [10]

$$Z_{k,N} = c_{k,N} e^{2\pi i k\tau} \frac{1}{\text{Vol U}(k)} \int d^{k^2} W^0 \, d^{4k^2} a \, d^{k^2} \phi \, d^{k^2} \bar\phi \, d^{4k^2} \mathcal{M} \, d^{4k^2} \zeta \times \\ \times \left(\det{}_{2k} W \det{}_k \bar\phi^2\right)^{N-2k} e^{-S_{k,N}} , \tag{3.6}$$

with

$$\begin{aligned} S_{k,N} &= -4\pi i \operatorname{tr}_k \bar\phi \Lambda^{12} + \operatorname{tr}_k \left(\bar\phi L \phi + \phi L \bar\phi\right), \\ \Lambda^{\dot A\dot B} &= \frac{1}{2\sqrt{2}} \left(\zeta^{[\dot A} W \zeta^{\dot B]} + [a^{\alpha\dot\alpha}, \mathcal{M}^{[\dot A}_{\alpha}]\zeta^{\dot B]}_{\dot\alpha} + \mathcal{M}^{\alpha[\dot A}\mathcal{M}^{\dot B]}_{\alpha}\right), \end{aligned} \tag{3.7}$$

and $c_{k,N} = c'_{k,N}(-2\pi^2)^{k(N-2k)}\pi^{4k^2}$. Here and in the following we denote by $\Phi^{[AB]} \equiv \Phi^{AB} - \Phi^{BA}$ the antisymmetrization in the $\mathrm{SU}(4)_{\mathcal{R}}$ indices.

We would like now to show that all dependence on the center of mass degrees of freedom in (3.6) factorizes in an $AdS_5 \times S^1$ term already at finite N. Inspired by large-N manipulations we split each U(k) adjoint V-field into its trace and traceless part

$$V = v_0 \mathbf{1}_{k\times k} + \hat v \,. \tag{3.8}$$

At this point it is convenient to rescale the various traceless components of the fields in the following way

$$\begin{aligned} a_m &= -x_m + \rho \hat a_m \,, \\ \phi &= r e^{i\vartheta}\left(1 + \hat\phi\right), \\ \bar\phi &= r e^{-i\vartheta}\left(1 + \hat{\bar\phi}\right), \\ W_0 &= \rho^2\left(1 + \hat W_0\right), \\ \zeta^{\dot A}_{\dot\alpha} &= \bar\eta^{\dot A}_{\dot\alpha} + \rho^{-1/2} e^{i\vartheta/2} \hat\zeta^{\dot A}_{\dot\alpha} \,, \\ \mathcal{M}^{\dot A}_{\alpha} &= \xi^{\dot A}_{\alpha} + \rho \hat a_{\alpha\dot\alpha} \bar\eta^{\dot\alpha\dot A} + \rho^{1/2} e^{i\vartheta/2} \hat{\mathcal{M}}^{\dot A}_{\alpha} \,. \end{aligned} \tag{3.9}$$

From (3.9) it follows that the variables x_m, ρ^2, $\xi^{\dot A}_{\dot\alpha}$, $\bar\eta^{\dot A}_{\dot\alpha}$ are the trace components associated to the overall bosonic and fermionic translational/conformal zero modes, while r is the distance of the D-instanton probe from the D3-branes.[5] Our results can be expressed in terms of this quantity and of the adimensional variable $y = r\rho$. In fact, plugging (3.9) in (3.6) and taking care of the Jacobian involved in the rescalings,

[5]Actually $r = d/\alpha'$, where d is the distance from the D3-branes in the orbifold fixed plane.

one can easily check that all dependence in the center of mass variables r, x_m, ϑ factors out in a "deformed" $AdS_5 \times S^1$ factor

$$
\begin{aligned}
Z_{k,N} &= d_{k,N} \int d^4x_m\, r^3 dr d\vartheta\, e^{4i\vartheta}\, e^{-2ikN\vartheta} r^{-2kN} e^{2\pi ik\tau_0}\, d^4\xi\, d^4\bar{\eta} \\
&= d_{k,N} \int d^4x_m\, r^3 dr d\vartheta\, e^{4i\vartheta}\, e^{-2ikN\vartheta} e^{2\pi ik\tau(r)}\, d^4\xi\, d^4\bar{\eta}\,.
\end{aligned}
\tag{3.10}
$$

Remarkably the entire deviation from the $AdS_5 \times S^1$ measure, which, up to numerical coefficients, is given by the factor $d^4x_m\, r^3 dr d\vartheta$ in (3.10),[6] is reabsorbed in the running coupling constant

$$
2\pi i\tau(r) \equiv 2\pi i\theta - \frac{8\pi^2}{g(r)^2} = 2\pi i\theta - \frac{8\pi^2}{g_0^2} - 2N\ln(r) = 2\pi i\tau_0 - 2N\ln(r)\,. \tag{3.11}
$$

In (3.11) we identify the AdS radial coordinate r with the dynamical energy scale μ in the SYM_4 gauge theory and read from the term multiplying the logarithm the first coefficient in the expansion of the β function

$$
\beta \equiv r\frac{d}{dr}g(r) = -\frac{b_1 g^3}{16\pi^2} + \cdots\,. \tag{3.12}
$$

with $b_1 = 2N_c - N_F = 2N$, where N_c, N_F are the number of colors and of the fundamental hypermultiplets, respectively.

Throughout this paper we employ units for which $\alpha' = 1$ and choose to work with the dimensioned instanton measure $Z_{k,N}$ rather than with the physical measure. The two differ by a numerical coefficient (see [7]) and a cut-off Λ dependent factor. Bringing back dimensions in (2.9) and (2.10) (recalling that $g_0 \sim (\alpha')^{-1}$) one can easily check that the dimensionless combination is precisely $\Lambda^{2kN} Z_{k,N}$. The appearance of the cut-off dependent factor in the normalized instanton measure can be rigorously derived from a Pauli-Villars regularization of the fluctuation determinants around the instanton background, see [29] for a complete discussion on this point. The net result of this normalization is to replace r in (3.11) by r/Λ, leading to the expected formula for the running coupling constant.

The extra phase factor $e^{-2i(kN-2)\vartheta}$ in (3.10) implements the selection rule imposed by the presence of the chiral anomaly. Indeed, taking into account that each insertion of an extra fermion, besides the eight associated to conformal and translational zero modes, carries a power of $e^{i\vartheta/2}$, one needs precisely $4(kN-2)$ insertions in order to cancel out the extra phase factor. This means that a non-trivial correlation function should involve a total number of $n = 4kN$ fermionic insertions, as the usual selection rule of pure $\mathcal{N} = 2$ SYM requires. For example, it would be interesting to consider the insertion of single or multi-trace composite operators of scalar fields, which have been recently shown not to get perturbative anomalous dimensions in $\mathcal{N} = 2$ theories [28].

[6] We use coordinates [1] where the $AdS_5 \times S_1$ metric reads $ds^2 = \frac{r^2}{R^2}dx_m^2 + \frac{R^2}{r^2}dr^2 + R^2 d\vartheta^2$ with $R^2 = \sqrt{4\pi g_s N}$, g_s being the string coupling constant.

Finally, the $d_{k,N}$ coefficients are given by the $\mathrm{SU}(k)$ integrals

$$
\begin{aligned}
d_{k,N} = c_{k,N} \frac{1}{\mathrm{Vol}\ \mathrm{SU}(k)} \int d\hat{W}^0\, d\hat{a}\, d\hat{\phi}\, d\hat{\bar{\phi}}\, d\hat{\mathcal{M}}\, d\hat{\zeta}\, dy y^{4kN-2k^2-5} \times \\
\times \left(\mathrm{det}_{2k} \widetilde{W} \, \mathrm{det}_k \, \tilde{\bar{\phi}}^2\right)^{N-2k} e^{-S(y)} \, ,
\end{aligned}
\tag{3.13}
$$

where we have introduced the compact tilded notation for the fields $\widetilde{W} = 1 + \hat{W}$ and $\tilde{\phi} = 1 + \hat{\phi}$. The action appearing in (3.13) reads

$$
S(y) = -4\pi i y \, \mathrm{tr}_k \, \tilde{\bar{\phi}} \tilde{\Lambda}^{12} - y^2 \mathrm{tr}_k \left(\tilde{\phi} \tilde{L} \tilde{\bar{\phi}} + \tilde{\bar{\phi}} \tilde{L} \tilde{\phi} \right) , \tag{3.14}
$$

with $\tilde{\Lambda}^{12}$ given by the same expression (3.7) as before but now in terms of the $\mathrm{SU}(k)$ components of the fermionic zero-modes and of the tilded field $\widetilde{W}$. For our purposes it will not be important to compute these coefficients, since they do not involve powers of r and therefore will not affect the associated geometry in (3.10).

3.2 Adding regular branes

In this subsection we extend the previous results to configurations involving both fractional and regular D3-branes moving on $\mathbb{R}^4/\mathbb{Z}_p$. More general orbifold projections $\mathbb{R}^6/\mathbb{Z}_p$ [13] are specified by four integers q_A such that $q_1 + q_2 + q_3 + q_4 = 0 \mod p$ and lead to $\mathcal{N} = 1, 0$ supersymmetric gauge theories depending on whether $\mathbb{Z}_p$ is embedded in an $\mathrm{SU}(3)$ or $\mathrm{SU}(4)$ subgroup of $\mathrm{SO}(6)_{\mathcal{R}} \cong \mathrm{SU}(4)_{\mathcal{R}}$, respectively. The former case corresponds to $q_A \neq 0$ for three values of the index A, while the latter to $q_A \neq 0$ for all values of A. Fractional instanton in these less supersymmetric theories are constrained to live on the boundary of the AdS_5 space[7] and therefore test only the less interesting four-dimensional flat geometry. This is the main reason why we restrict our analysis to the $\mathcal{N} = 2$ case even if most of our manipulations apply in the $\mathcal{N} = 0, 1$ contexts. We will follow closely [11], but focusing as in the previous section on non-conformal brane configurations. We denote by N_q with $q = 0, 1, \ldots, p-1$ the number of D3-branes transforming in the qth-representation of $\mathbb{Z}_p$. A regular brane is defined by a set of p D3-branes, one of each type, and therefore the general configuration can be thought as built out of $n = \min_q \{N_q\}$ regular branes plus $N_q - n$ fractional branes of the q-type. The example of pure $\mathcal{N} = 2$ studied in the previous section corresponds to the case of no regular branes $n = N_q = 0$ for $q > 0$ with $N_0 = N$ fractional branes of the 0th-type. In a similar way we denote by k_q the number of D(-1) instantons of a given type. We consider first the case where the D3-brane geometry is probed by fractional instantons (k_q generic). The modifications to the case where regular instantons $k_0 = k_2 = \cdots = k_{p-1} = K/p$ are used as test charges will be commented only at the end of this section.

[7] In the previous section and all the time along this paper we studied the case $\mathbb{R}^6/\mathbb{Z}_p = \mathbb{R}^4/\mathbb{Z}_p \times \mathbb{R}^2$ in which fractional instantons are allowed to move in the directions 3,6 of the transverse space. Here we refer to the case where the transverse space is $\mathbb{R}^6/\mathbb{Z}_p$ and fractional instantons live at the fixed point $r = 0$.

The analysis of the resulting gauge theory follows closely the one in the previous section but taking care of the fact that now the $\mathbb{Z}_p$ projection acts on both Chan-Paton and $\mathrm{SU}(4)_{\mathcal{R}}$ indices. The precise action on the various D(−1)–D3 fields is given by

$$
\begin{aligned}
w_{\dot\alpha} &= \gamma_N\, w_{\dot\alpha}\, \gamma_K^{-1}\,, & \mu^A &= e^{2\pi i \frac{q_A}{p}} \gamma_N\, \mu^A\, \gamma_K^{-1}\,,\\
\bar w^{\dot\alpha} &= \gamma_K\, \bar w^{\dot\alpha}\, \gamma_N^{-1}\,, & \bar\mu^A &= e^{2\pi i \frac{q_A}{p}} \gamma_K\, \bar\mu^A\, \gamma_N^{-1}\,,\\
a_{\alpha\dot\alpha} &= \gamma_K\, a_{\alpha\dot\alpha}\, \gamma_K^{-1}\,, & \mathcal{M}^A_\alpha &= e^{2\pi i \frac{q_A}{p}} \gamma_K\, \mathcal{M}^A_\alpha\, \gamma_K^{-1}\,,\\
\chi_{AB} &= e^{2\pi i \frac{q_A+q_B}{p}} \gamma_K\, \chi_{AB}\, \gamma_K^{-1}\,, & D^c &= \gamma_K\, D^c\, \gamma_K^{-1}\,, \qquad \lambda^A_{\dot\alpha} = e^{2\pi i \frac{q_A}{p}} \gamma_K\, \lambda^A_{\dot\alpha}\, \gamma_K^{-1}\,,
\end{aligned}
$$

where $q_1 = q_2 = 0, q_3 = -q_4 = 1$, and γ_K, γ_N are $K \times K$ and $N \times N$ matrices (with $N = \sum_q N_q$ and $K = \sum_q k_q$) realizing the orbifold group action on the Chan-Paton indices. The surviving components can be written as

$$
\begin{aligned}
w^q_{\dot\alpha} &= w_{\dot\alpha u_q i_q}\,, & \mu^A_q &= \mu^A_{u_q i_{q+q_A}}\,,\\
\bar w^{\dot\alpha}_q &= \bar w_{\dot\alpha i_q u_q}\,, & \bar\mu^A_q &= \bar\mu^A_{i_{q+q_A} u_q}\,,\\
a^{\alpha\dot\alpha}_q &= a^{\alpha\dot\alpha}_{i_q j_q}\,, & \mathcal{M}^{A\alpha}_q &= \mathcal{M}^{A\alpha}_{i_{q+q_A} j_q}\,,\\
\chi^{AB}_q &= \chi^{AB}_{i_{q+q_A} j_{q-q_B}}\,, & D^c_q &= D^c_{i_q j_q}\,, \qquad \lambda^{A\dot\alpha}_q = \lambda^{A\dot\alpha}_{i_{q+q_A} j_q}\,,
\end{aligned}
\tag{3.15}
$$

where $q = 0, 1, \ldots, p-1$, $i_q = 1, \ldots, k_q$, $u_q = 1, \ldots, N_q$ are the block indices.[8] All sums in q are from now on understood modulo p.

The result (3.15) is derived from an iterative application of the Mac-Kay correspondence. We start by defining $K(N)$-dimensional vector spaces $V(W)$. Under the action of $\Gamma = \mathbb{Z}_p$ they decompose as $V = \sum_q k_q V_q$ ($W = \sum_q N_q W_q$) where $\{k_q\}(\{N_q\})$ labels the number of D(−1)–(D3)-branes transforming in the qth-irreducible representation R_q of $\mathbb{Z}_p$ and $V_q(W_q)$ are one-dimensional vector spaces associated to this irreducible representation. Open string modes can be thought as homomorphisms (or endomorphisms) between the real vector spaces V, W. In addition to the above decomposition in the Chan-Paton space, open string modes carrying a $\bar A = 3, 4$ index transform in the so called regular defining representation Q of $\mathbb{Z}_p$. The Mac-Kay correspondence states that the representation $R_q \otimes Q$ decompose under the action of a discrete subgroup Γ of SU(2) as

$$
R_q \otimes Q = \oplus_r A_{qr} R_r \tag{3.16}
$$

with $A_{qr} = 2\delta_{qr} - \tilde C_{qr}$, $\tilde C_{qr}$ being the extended Cartan matrix of the A-D-E algebra associated to Γ. In the present situation $\Gamma = \mathbb{Z}_p$ and $A_{qr} = \delta_{q,r+1} + \delta_{q,r-1}$. Collecting together $\dot A = 1, 2$ and $\bar A = 3, 4$ indices in the "A" index, we can write (3.16) as

[8] More precisely the indices in (3.15) run as $i_1 = 1, \ldots, k_1, i_2 = k_1 + 1, \ldots, k_1 + k_2$ and so on. We adopt the simplifying notation where the subscripts "q" indicates the matrix blocks from which i_q, u_q start to count.

$R_q \otimes Q_A = \delta_{q,q+q_A} R_q$ with $A = 1, \ldots, 4$. The Γ invariant components can then be easily found using the Schur' s lemma $\mathrm{Hom}(R_q, R_r)_\Gamma = \delta_{qr}$. Let us illustrate how this works for the fields $(\chi_{AB}, \lambda^A_{\dot\alpha}, D^c)$ in the vector multiplet:

$$\begin{aligned} \chi_{AB} : \mathrm{Hom}\left(Q_A \otimes V, \bar{Q}_B \otimes V\right)_\Gamma &= \mathrm{Hom}\left(R^{k_{q+q_A}}, R^{k_{q-q_B}}\right), \\ \lambda^A_{\dot\alpha} : \mathrm{Hom}_{\dot\alpha}\left(Q_A \otimes V, V\right)_\Gamma &= \mathrm{Hom}_{\dot\alpha}\left(R^{k_{q+q_A}}, R^{k_q}\right), \\ D^c : \mathrm{Hom}_c\left(V, V\right)_\Gamma &= \mathrm{Hom}_c\left(R^{k_q}, R^{k_q}\right). \end{aligned} \tag{3.17}$$

It is a straightforward exercise to extend this analysis to the remaining fields of the D(−1)–D3 lagrangian, leading finally to the matrix block form (3.15).

Coming back to (3.15) one notice that after the Γ projection the D(−1) gauge group is reduced to $\prod_q \mathrm{U}(k_q)$.[9] A similar projection on the D3-D3 strings shows that the four-dimensional gauge group is reduced to $\prod_q \mathrm{U}(N_q)$, and that the field theory spectrum is given by vector multiplets in the adjoint representation of the product gauge group and matter hypermultiplets in the bifundamental representations $(\bar{N}_q, N_{q+q_A})$.

Mutatis mutandis we follow the strategy of the previous subsection, see (3.3), (3.4) and (3.5). The partition function (3.6) generalizes to

$$\begin{aligned} \mathcal{Z}_{k,N} = \prod_q \Bigg\{ c_{k_q,N_q} e^{2\pi i k_q \tau_q} \frac{1}{\mathrm{Vol}\,\mathrm{U}(k_q)} \int d^{k_q^2} W^0\, d^{4k_q^2} a\, d^{2k_q\left(4\hat{k}_q - k_q\right)} \chi\, d^{8k_q\hat{k}_q} \mathcal{M}\, d^{8k_q\hat{k}_q} \zeta \,\times \\ \times \left(\det_{2k_q} W \det_{4\hat{k}_q} \chi\right)^{N_q - 2k_q} \Bigg\} \exp\left[4\pi i\, \mathrm{tr}_k\, \chi_{AB} \Lambda^{AB} - \mathrm{tr}_k\, \chi_a L \chi_a\right]. \end{aligned} \tag{3.18}$$

with $\hat{k}_q = \frac{1}{4} \sum_{A=1}^4 k_{q+q_A}$.

The $\dot{A}, \dot{B} = 1, 2$ components of the fields in (3.15) can be split exactly as in the previous formula (3.9) into their trace and traceless part, where now the instanton center of mass degrees of freedom clearly belong to the block diagonal components in (3.15), while the $\mathrm{SU}(k_q)$ fields are given in terms of still block diagonal but traceless $q \times q$ matrices.

The remaining $\bar{A}, \bar{B} = 3, 4$ components in (3.15) are instead off-diagonal, and can be conveniently rescaled as

$$\begin{aligned} \phi^s &= r e^{-i\vartheta} \hat{\phi}^s, \\ \bar{\phi}^s &= r e^{i\vartheta} \hat{\bar{\phi}}^s, \\ \zeta^{\bar{A}}_{\dot\alpha} &= \rho^{-1/2} e^{-i\frac{\vartheta}{2}} \hat{\zeta}^{\bar{A}}_{\dot\alpha}, \\ \mathcal{M}^{\bar{A}}_\alpha &= \rho^{1/2} e^{-i\frac{\vartheta}{2}} \hat{\mathcal{M}}^{\bar{A}}_\alpha, \end{aligned} \tag{3.19}$$

where by

$$\phi^s = \frac{1}{\sqrt{2}}\left(-\chi_s + i\chi_{3+s}\right), \qquad s = 1, 2 \tag{3.20}$$

[9] The existence of this "auxiliary" $\prod_q \mathrm{U}(k_q)$ corresponds to the internal symmetries of the ADHM construction. See [19, 20] for a more detailed discussion of its role.

we indicate the two complex coordinates which span the $\mathbb{R}^4/\mathbb{Z}_p$ space orthogonal both to the D3-brane and to the orbifold fixed plane.

Plugging in (3.18) one is left with the final result[10]

$$\begin{aligned}\mathcal{Z}_{K,N} &= D_{K,N}\int d^4x_m\, r^3 dr d\vartheta\, e^{4i\vartheta}\, \left(re^{i\vartheta}\right)^{-4N_q(k_q-\hat{k}_q)}\, e^{2\pi i\, k_q\tau_q^0}\, d^4\xi\, d^4\bar{\eta} \\ &= D_{K,N}\int d^4x_m\, r^3 dr d\vartheta\, e^{4i\vartheta}\, e^{-4i\vartheta N_q\left(k_q-\hat{k}_q\right)} e^{2\pi i k_q\tau_q(r)}\, d^4\xi\, d^4\bar{\eta}\,, \end{aligned} \tag{3.21}$$

where $D_{K,N}$ is a numerical coefficient given again in terms of integrals over the $\mathrm{SU}(k_q)$ fields in (3.18).

We can see from (3.21) that the expected $AdS_5\times S^1$ volume form corresponding to the multi-instanton center of mass degrees of freedom factors out while the additional powers of r are reabsorbed in the gauge coupling constants $\tau_q(r)$ associated to the q-th gauge group

$$\begin{aligned}2\pi i k_q\tau_q(r) &\equiv 2\pi i k_q\theta_q - \frac{8\pi^2 k_q}{g_q^2(r)} \\ &= 2\pi i k_q\theta_q - \frac{8\pi^2 k_q}{g_q^2} - 4N_q(k_q-\hat{k}_q)\,\ln(r)\,. \end{aligned} \tag{3.22}$$

We can now check (3.22) against the running of the coupling constants computed with field theory methods. As anticipated, the field content of our SYM_4 can be organized in organized in terms of $\mathcal{N}=2$ supermultiplets as

$$\begin{aligned}\text{Vector multiplets}: &\quad \text{Adjoint of } \prod_q \mathrm{U}(N_q)\,, \\ \text{Hypermultiplets}: &\quad (\bar{N}_q, N_{q+1}) + (\bar{N}_q, N_{q-1})\,. \end{aligned} \tag{3.23}$$

For each $\mathrm{U}(N_q)$ factor we then have one vector multiplet and $N_F^{(q)} = N_{q+1}+N_{q-1}$ fundamental hypermultiplets. The first coefficient in the expansion of the beta function is given by

$$b_1^{(q)} = 2N_q - N_F^{(q)} = 2N_q - N_{q+1} - N_{q-1}\,. \tag{3.24}$$

The expression (3.24) agrees with the D-instanton result (3.22) noticing that

$$k_q b_1^{(q)} = k_q\left(2N_q - N_{q+1} - N_{q-1}\right) = 4N_q\left(k_q - \hat{k}_q\right). \tag{3.25}$$

We remark that, as for the pure $\mathcal{N}=2$ case of the previous section, we do not need an explicit expression for the $D_{K,N}$ coefficients in (3.21) for our present purposes, since they do not affect the resulting geometry. Nonetheless, by studying their behaviour in the large-N limit, which is amenable to a saddle point expansion, we can make contact with the more quantitative results of [10]. This corresponds to the case

[10] The sum over repeated q indices is always understood.

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$k_0 = K, k_1 = \cdots = k_{p-1} = 0$ and $N_0 = N_1 = \cdots = N_{p-1} = N$, which, accordingly to (3.23), describe charge K instantons in a $\mathcal{N} = 2$ U(N) theory with $N_F = 2N$ fundamental hypermultiplets [10].

The relevant large-N effective action for the bosonic fields can be obtained along the same lines of [10]. In particular, it can be shown that the dominant saddle point contribution in the large-N limit is the maximally degenerated solution, which corresponds to vanishing hatted fields in (3.9) and (3.19). Plugging this solution in the multi-instanton action and exponentiating the powers of y which appear in the coefficient $D_{K,N}$, one is left with the effective action

$$S_{\text{eff}} \sim 2KN\left(y^2 - 2\ln y\right), \tag{3.26}$$

with a saddle point at $y = 1$, i.e. $r = \rho^{-1}$ [7, 10].

The SU(K) fluctuations around this solution can be treated in the large-N limit as in [10]; in particular, it turns out that their dynamics is governed by a $0+0$-dimensional gauge theory obtained from the dimensional reduction of a $\mathcal{N} = 1$ SU(K) gauge theory in $D = 6$. Thus in this case the large-N limit of the $D_{K,N}$ coefficients is proportional to the partition function $\hat{\mathcal{Z}}_K^{(6)}$ of the above mentioned $0+0$-dimensional gauge theory.

Finally let us consider the case where the D3-brane geometry is probed by regular D-instantons $k_0 = k_1 = \cdots = k_{p-1} = K/p$. First of all notice that while fractional instantons were obliged to live in the orbifold plane $\phi_s = 0$,[11] regular D-instantons come together with their images and can roam in the entire transverse space $\mathbb{R}^6$. Then, the ansatz (3.19) should be improved in order to take into account the new bosonic and fermionic zero modes associated to the translational modes along the orbifolded directions. The r dependence of the instanton measure in this case follows directly from (3.21) but the angular dependence, which is now on $S_5/\mathbb{Z}_p$, requires trickier manipulations due to the non-commutativity of the SO(6)$_{\mathcal{R}}$ isometry group. The new ansatz can be written as

$$\begin{aligned}
\chi_a &= r g_{ab}\left(\mathbf{e}_b + \hat{\chi}_b\right), \\
\zeta^A_{\dot\alpha} &= \bar{\eta}^A_{\dot\alpha} + \rho^{-\frac{1}{2}}\hat{\zeta}^A_{\dot\alpha}, \\
\mathcal{M}^A_\alpha &= \xi^A_\alpha + \rho\hat{a}_{\alpha\dot\alpha}\bar{\eta}_{\dot\alpha A} + \rho^{1/2}\hat{\mathcal{M}}^A_\alpha,
\end{aligned} \tag{3.27}$$

where we denoted by $r g_{ab}\mathbf{e}_b$ the multi-instanton center of mass degrees of freedom, which are now no longer diagonal. Instead, they are built out of square identity blocks:

$$\begin{aligned}
\mathbf{e}_{AB} &\equiv \frac{1}{\sqrt{8}}\mathbf{e}_b\Sigma^b_{AB} = e_{AB}\delta_{i_{q+q_A}j_{q-q_B}}, \\
\bar{\eta}^A_{\dot\alpha} &\equiv \bar{\eta}^A_{\dot\alpha}\delta_{i_q j_{q+q_A}}, \\
\xi^A_{\dot\alpha} &\equiv \xi^A_{\dot\alpha}\delta_{i_q j_{q+q_A}},
\end{aligned} \tag{3.28}$$

[11] We recall that this is the $\mathbb{R}^2$ part of the transverse space not acted upon by $\mathbb{Z}_p$.

suitably disposed inside the $K \times K$ matrix in a way which preserves the Γ-invariance [11]. We have introduced in (3.27) a reference unit vector $e_b = -\bar{\Sigma}_b^{AB} e_{AB}/\sqrt{2}$ which we fix to lie along, say, the first axis, $e_b = \delta_{b1}$. Moreover, g_{ab} is a matrix in the coset group $\mathrm{SO}(6)/\mathrm{SO}(5)$ parametrizing the orientation with respect to e_b of the multi-instanton center of mass, i.e. a position in S^5. Now, by plugging (3.27) in the action appearing in (3.18) and reabsorbing the remnant dependence on g_{ab} of the Yukawa term by a suitable redefinition of the Clifford gamma matrices, one can easily check the factorization of a center of mass term $AdS_5 \times S^5/\mathbb{Z}_p$. The global $\mathbb{Z}_p$ identification is the result of a remnant discrete symmetry $\mathbf{e}_{AB} \sim e^{2\pi i(q_A+q_B)/p}\mathbf{e}_{AB}$ [10].

Notice that there is no deformation of the AdS geometry. Indeed, in this case $k_0 = k_1 = \cdots = k_{p-1}$, which implies $\hat{k}_q = k_q$, and from (3.25) one can immediately see that no additional powers of r come into the measure (3.21). We conclude that regular instantons probe an $AdS_5 \times S^5/\mathbb{Z}_p$ (just as in the conformal case [10]) even in the presence of fractional D3-branes, i.e. non-conformal geometries. This is again in agreement with our field theory expectations since the overall sum of the gauge coupling constants, related to the ten-dimensional type-IIB coupling constant, does not run.

4. D-instanton actions and topological field theories

It is very useful for practical computations to write down a topological version of the multi-instanton action we discussed in the previous section. In fact the presence in this case of a scalar supersymmetry Q allows to display interesting localization properties of the functional integral. Moreover, the Ward identities associated to Q allow in some cases to easily rule out the contribution of Q-exact terms in the action [20, 21, 26]. In [20] this task was accomplished starting from the topologically twisted version of $\mathcal{N} = 2$ SYM_4 in $\mathbb{R}^4$. In this model, the functional integration over the fields can be done exactly, and localizes the partition function on the instanton moduli space [30]; correspondingly, the dynamics of the multi-instanton collective coordinates in presence of a non-trivial vacuum expectation value for the scalar field is described by a cohomological action on the ADHM variables.

The introduction of auxiliary fields and the D-brane interpretation leading to (2.10), in conjunction with the orbifold projection, allows a more general formulation of the above cited results. It is in fact now possible to twist the theory at the level of the parameters describing the moduli space and to show that the theory is given by the implementation of the ADHM constraints *à la* BRST. All previous results are instantly recovered: the results with a vacuum expectation value for the scalar in the theory are obtained by shifting the χ_a fields surviving the orbifold projection. Also the topological theory obtained in the $\mathcal{N} = 4$ case [21] matches with our results after the orbifold projection.

We remark that the partition function we considered in the previous section is multiplied by the eight fermionic zero modes $(\xi^{\dot{A}}_{\alpha}, \bar{\eta}^{\dot{A}}_{\dot{\alpha}})$ associated to the translational and superconformal invariance, such that, in order to get well defined physical quantities, one is forced to insert suitable observables. The presence of a vacuum expectation value for the scalar fields breaks superconformal invariance and makes the associated fermionic zero-modes $\bar{\eta}^{\dot{A}}_{\dot{\alpha}}$ to appear in the multi-instanton action. Correspondingly, the partition function of the *centered* moduli space[12] becomes itself a non-trivial observable. However, gauge theories with a non-trivial v.e.v. correspond to less symmetric D-branes configurations, making the explicit computations much more involved. In this case, the localization properties which show up in the topological formulation could give useful simplifications. On the other hand, this formulation could be used to identify a set of non-trivial observables useful to analyse finite-N effects also in the conformal phase. We finally remark that another way to break superconformal invariance is to consider a non-commutative geometry background; as we will see, also this model can be easily included in a topological framework.

The $\mathcal{N} = 2$ D-instanton action can be written in the cohomological theory set up by resorting to the standard twisting procedure [30]. In fact, as we discussed in section 3, after the orbifold projection (3.1), the field content of the theory can be read from (2.3) by restricting A, B to the $\dot{A}, \dot{B}$ indices labelling the fundamental representation of the $\mathcal{N} = 2$ automorphism group $\mathrm{SU}(2)_{\dot{A}}$. We then redefine the four-dimensional rotation group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ acting on the projected fields as $\mathrm{SU}(2)_L \times \mathrm{SU}(2)'_R$, where $\mathrm{SU}(2)'_R$ is the diagonal subgroup of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_{\dot{A}}$. We remark that on a flat manifold this simply amounts to a redefinition of variables, such that the conventional and the twisted formulation are completely equivalent.

In view of the above considerations, we also consider the possibility of turning on a vacuum expectation value v for the adjoint scalar and a background B-field along the D3-brane directions. The former correspond to a separation between the D3-branes and is easily realized by replacing ϕ by $\phi + v$ in all previous formulas.[13] The latter can be incorporated by adding a Fayet-Iliopoulos term ζ^c to the D(-1)–D3 lagrangian and give rise to a theory on a non-commutative space-time [31]

$$[x_m, x_n] = -i\zeta^c \bar{\eta}^c_{mn} \,. \tag{4.1}$$

Accordingly, the bosonic ADHM constraints are deformed by the parameter ζ^c [32], such that we have, in terms of the twisted variables (where $\dot{A}$ is identified with $\dot{\alpha}$)

$$\begin{aligned} \bar{w}^{\dot{\alpha}} \mu_{\dot{\alpha}} - \bar{\mu}^{\dot{\alpha}} w_{\dot{\alpha}} - 2\,[a_n, \mathcal{M}_n] &= 0\,, \\ \bar{w}\tau^c \mu + \bar{\mu}\tau^c w - 2i\bar{\eta}^c_{mn}\,[a_m, \mathcal{M}_n] &= 0\,, \\ \bar{w}\tau^c w - i\bar{\eta}^c_{mn}\,[a'_m, a'_n] &= \zeta^c \,. \end{aligned} \tag{4.2}$$

[12]We recall that the centered moduli space is defined as the quotient of the usual moduli space with respect to the instanton center of mass x_m and the associated four fermionic zero-modes $\xi^{\dot{A}}_{\alpha}$.

[13]For simplicity we restrict v to lie in the Cartan subalgebra of the gauge group.

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To these constraints we associate the doublets of auxiliary fields (λ_c, D_c) and $(\bar\phi, \eta)$, transforming under the scalar supercharge Q as

$$\begin{aligned} Q\,\lambda_c &= D_c\,, & Q\,D_c &= [\phi, \lambda_c]\,, \\ Q\,\bar\phi &= \eta\,, & Q\,\eta &= [\phi, \bar\phi]\,, \\ Q\,\phi &= 0\,, & Q\,v &= Q\bar v = 0\,, \\ Q\,\zeta^c &= 0\,. \end{aligned} \tag{4.3}$$

where ϕ is the generator of $\mathrm{U}(k)$ transformations under which (4.2) are invariant.

The cohomological action which implements the constraints (4.2) reads

$$\begin{aligned} S = \mathrm{tr}_k\, Q \biggl[&(\bar\phi + \bar v)\left(\bar w^{\dot\alpha}\mu_{\dot\alpha} - \bar\mu^{\dot\alpha} w_{\dot\alpha} - 2\,[a_n, \mathcal{M}_n]\right) + \frac{1}{g_0^2}\eta\,[\phi, \bar\phi] + \\ &+ \lambda_c \left(\bar w \tau^c w - i\bar\eta^c_{mn}\,[a_m, a_n] - \zeta^c\right) - \frac{1}{g_0^2}\lambda_c D_c \biggr]\,, \end{aligned} \tag{4.4}$$

with

$$\begin{aligned} Q\,a_n &= \mathcal{M}_n\,, & Q\,\mathcal{M}_n &= [\phi, a_m]\,, \\ Q\,w_{\dot\alpha} &= \mu_{\dot\alpha}\,, & Q\,\mu_{\dot\alpha} &= -w_{\dot\alpha}(\phi + v)\,, \\ Q\,\bar w^{\dot\alpha} &= \bar\mu^{\dot\alpha}\,, & Q\,\bar\mu^{\dot\alpha} &= (\phi + v)\bar w^{\dot\alpha}\,. \end{aligned} \tag{4.5}$$

We observe that in this context g_0^2 can be understood as a "gauge-fixing" parameter, which can be continuously deformed without affecting the results as long as the Q-invariant physical observables we are computing are well defined. In particular, we can send $g_0^2 \to \infty$, such that the ADHM constraints (4.2) are implemented as Dirac delta-functions instead to be spread out as Gaussian weights in the functional integral [21, 26]. As we have seen in the previous sections, this significantly simplifies the explicit computations.

In the presence of N_F fundamental hypermultiplets, the multi-instanton action gets another term [10] which can be written as

$$S_{\mathrm{hyp}} = -Q\mathrm{tr}_k \left[\tilde h \mathcal{K} + \tilde{\mathcal{K}} h\right]\,, \tag{4.6}$$

where $(\mathcal{K}, \tilde{\mathcal{K}})$ are, respectively $k \times N_F$ and $N_F \times k$ matrices denoting the collective fermionic coordinates for the fundamental matter fields and $(h, \tilde h)$ are bosonic auxiliary variables, transforming as

$$\begin{aligned} Q\,\mathcal{K} &= h\,, & Q\,h &= \phi\mathcal{K}\,, \\ Q\,\tilde{\mathcal{K}} &= \tilde h\,, & Q\,\tilde h &= -\tilde{\mathcal{K}}\phi\,. \end{aligned} \tag{4.7}$$

By acting with Q on the l.h.s. of (4.4) and splitting the action as in (2.9) we get, after integration on the auxiliary variables $(h, \tilde h)$

$$S_G = \mathrm{tr}_k \left([\phi, \bar\phi]^2 - \eta[\phi, \eta] + \lambda_c[\phi, \lambda_c] - D^c D^c\right)\,,$$

$$
\begin{aligned}
S_K &= \mathrm{tr}_k \left(-2\,[\phi, a_n]\,[\bar\phi, a_n] + \left[(\phi+v)(\bar\phi+\bar v) + \mathrm{h.c.}\right] \bar w^{\dot\alpha} w_{\dot\alpha} + \right.\\
&\quad \left. + 2\mathcal{M}_m[\bar\phi, \mathcal{M}_m] + 2(\bar\phi+\bar v)\bar\mu^{\dot\alpha}\mu_{\dot\alpha}\right),\\
S_D &= \mathrm{tr}_k \left[\eta\left(\bar w^{\dot\alpha}\mu_{\dot\alpha} - \bar\mu^{\dot\alpha} w_{\dot\alpha} - 2\,[a_n, \mathcal{M}_n]\right) - \lambda_c\left(\bar w\tau^c\mu + \bar\mu\tau^c w - 2i\bar\eta^c_{mn}\,[a_m, \mathcal{M}_n]\right) + \right.\\
&\quad \left. + D^c\left(\bar w\tau^c w - i\bar\eta^c_{mn}[a_m, a_n] - \zeta^c\right)\right],\\
S_{\mathrm{hyp}} &= \mathrm{tr}_k\left[2\tilde{\mathcal{K}}\phi\mathcal{K}\right]. \qquad (4.8)
\end{aligned}
$$

We remark that this action can also be obtained from dimensional reduction to $(0+0)$ dimensions of the two-dimensional $(0,4)$ supersymmetric Yang-Mills theory which describes the low-energy dynamics of a D1-D5 system wrapping a $\mathbb{R}^4/\mathbb{Z}_2$ space. In this case, the fermionic symmetry Q is given by a suitable combination of the supersymmetry charges [33].

Few comments are in order for the pure $\mathcal{N}=2$ theory. When $N_F = 0$, the action (4.8) coincides for $v = \zeta^c = 0$ with the multi-instanton action of the previous section 3.1 written in terms of the twisted variables

$$
\begin{aligned}
\lambda^{\dot\alpha}_{\dot A} &= \left(\sqrt{2}\pi i\right)^{1/2}\left(i\eta\sigma^4 + \lambda_c\sigma^c\right)^{\dot\alpha}_{\dot A},\\
\mathcal{M}^{\alpha\dot A} &= \left(\frac{\pi i}{\sqrt{2}}\right)^{1/2}\mathcal{M}_n\bar\sigma_n^{\alpha\dot A},\\
\mu^{\dot A} &= \left(\frac{\pi i}{\sqrt{2}}\right)^{1/2}\mu^{\dot A}. \qquad (4.9)
\end{aligned}
$$

Moreover, we recall that a cohomological action for the commutative case $\zeta^c = 0$ and in presence of a non-trivial v.e.v. v for the scalar field was obtained in [20] by resorting to the topologically twisted formulation of the $\mathcal{N}=2$ SYM theory. Once the ADHM constraints have been enforced by integration on the lagrangian multipliers (D^c, λ^c, η), the same multi-instanton action can be obtained from the S_K term in (4.8) by integrating on $(\phi, \bar\phi)$, which implements the equation of motion for the scalar field

$$
L\phi = [\mathcal{M}_m, \mathcal{M}_m] - \bar\mu^{\dot\alpha}\mu_{\dot\alpha}\,, \qquad (4.10)
$$

where the operator L was defined in (2.7). The scalar supersymmetry operator (4.5) coincides then with the covariant derivative on the moduli space defined in [20],[14]

$$
Q = s + \mathcal{C}\,, \qquad (4.11)
$$

where s and $\mathcal{C}$ are, respectively the exterior derivative and the $\mathrm{U}(k)$ connection on the ADHM moduli space, ϕ being the corresponding $\mathrm{U}(k)$ curvature.

In appendix B we show how our $\mathcal{N}=2$ topological theory can be obtained from that derived in [21] for the $\mathcal{N}=4$ case by using the same projection procedure discussed in section 3.

[14]actually, in [20] the $\mathrm{Sp}(1) \sim \mathrm{SU}(2)$ ADHM formalism has been adopted; hence, the ADHM auxiliary group is $\mathrm{O}(k)$. This makes to appear sligthly different notations for the explicit expression of the action and the connection, but obviously the geometrical interpretation holds unaltered.

5. Instantons on ALE spaces

Finally we would like to comment on the case in which the orbifold quotient is taken along the directions longitudinal to the D3-brane system. These directions form a $\mathbb{R}^4$ space acted upon by the Lorentz group $\mathrm{SO}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. This case has been extensively studied in [27] and D-instantons were shown to reproduce the Kronheimer-Nakajima construction of self-dual connections for gauge theories living on ALE spaces [34]. The projection is similar to the one performed in section 2 but now Γ is acting on the Lorentz indices α, β of $\mathrm{SU}(2)_L$.

There is however an important difference respect to our previous considerations of gauge instantons in $\mathbb{R}^4$. Indeed, noticing that the ALE space geometry support topologically non-trivial two cycles, the instanton solution is now classified by both the first and second Chern classes. More precisely one can associate [35] a tautological bundle $\mathcal{T}$ with fiber the regular representation R of Γ and base the ALE space itself. Under the action of Γ this tautological bundle admits a decomposition $\mathcal{T} = \sum_q \mathcal{T}_q \otimes R_q$ with R_q $(q = 0, 1, \ldots, p-1)$ the irreducible representation of Γ. The first Chern Class $c_1(\mathcal{T}_q)$ of the $\mathcal{T}_q$ bundles, $q \neq 0$ ($c_1(\mathcal{T}_0) = 0$), form a basis of the second cohomology group and satisfy

$$\int_{\mathrm{ALE}} c_1\left(\mathcal{T}_q\right) \wedge c_1\left(\mathcal{T}_{q'}\right) = \left(C^{-1}\right)_{qq'} \tag{5.1}$$

with C^{-1} the inverse of the Cartan matrix of the unextended Dynkin diagram. The restriction to the interesting case of instanton solutions with vanishing first Chern class impose, as we will see, strong constraints on the allowed instanton configurations $\{k_q\}$ for a given partition $\{N_q\}$. In the following we follow closely [35] whose notation we adapt to the computations carried out in the previous sections.

Once again the starting point is the $K = \sum_q k_q$ instanton instanton solution of $\mathcal{N} = 4$ SYM_4 on a group $\mathrm{U}(N)$, with $N = \sum_q N_q$. The projection (5.2) is now replaced by

$$\begin{aligned}
w_{\dot\alpha} &= \gamma_N\, w_{\dot\alpha}\, \gamma_K^{-1}\,, & \mu^A &= \gamma_N\, \mu^A\, \gamma_K^{-1}\,, \\
\bar w^{\dot\alpha} &= \gamma_K\, \bar w^{\dot\alpha}\, \gamma_N^{-1}\,, & \bar\mu^A &= \gamma_K\, \bar\mu^A\, \gamma_N^{-1}\,, \\
a_{\alpha\dot\alpha} &= e^{2\pi i\frac{q_\alpha}{p}}\gamma_K\, a_{\alpha\dot\alpha}\, \gamma_K^{-1}\,, & \mathcal{M}^A_\alpha &= e^{2\pi i\frac{q_\alpha}{p}}\gamma_K\, \mathcal{M}^A_\alpha\, \gamma_K^{-1}\,, \\
\chi_{AB} &= \gamma_K\, \chi_{AB}\, \gamma_K^{-1}\,, & D^c &= \gamma_K\, D^c\, \gamma_K^{-1}\,, \qquad \lambda^A_{\dot\alpha} = \gamma_K\, \lambda^A_{\dot\alpha}\, \gamma_K^{-1}
\end{aligned} \tag{5.2}$$

with $q_1 = -q_2 = 1$, and γ_K, γ_N the $K \times K$ and $N \times N$ matrices realizing the orbifold group action on the Chan-Paton indices. Notice that supersymmetry is preserved by this projection since Γ acts in the same way on the different components of a given supermultiplet.

The surviving components read

$$
\begin{aligned}
w^q_{\dot\alpha} &= w_{\dot\alpha i_q u_q}\,, & \mu^A_q &= \mu^A_{i_q u_q}\,, & &\\
a^{\alpha\dot\alpha}_q &= a^{\alpha\dot\alpha}_{i_{q+q_\alpha} j_q}\,, & \mathcal{M}^{A\alpha}_q &= \mathcal{M}^{A\alpha}_{i_{q+q_\alpha} j_q}\,, & &\\
\chi^{AB}_q &= \chi^{AB}_{i_q j_q}\,, & D^c_q &= D^c_{i_q j_q}\,, & \lambda^{A\dot\alpha}_q &= \lambda^{A\dot\alpha}_{i_q j_q}\,.
\end{aligned}
\tag{5.3}
$$

The moduli space of multi-instanton solutions and ADHM constraints can be described as before through (2.3) and (2.4) but now in terms of the invariant components (5.3). In particular the dimension of the moduli space is given by the total number of components in the first two lines of (5.3), minus the number of ADHM constraints (given by dim $D^c = 3k_q^2$ for each q), minus the dimension of the auxiliary gauge group $\prod_q \mathrm{U}(k_q)$:

$$
\begin{aligned}
\dim \mathcal{M}_B &= 4\sum_q \left(k_q N_q + \hat{k}_q k_q - k_q^2\right),\\
\dim \mathcal{M}_F &= 8\sum_q \left(k_q N_q + \hat{k}_q k_q - k_q^2\right),
\end{aligned}
\tag{5.4}
$$

where $\hat{k}_q = \frac{1}{2}\sum_\alpha k_{q+q_\alpha}$. The resulting dimensions in (5.4) are in agreement with [35].

The definition of the ADHM matrices (2.1) and the ADHM ansatz (2.5) require instead some generalization in order to encompass the non-triviality of the ALE geometry. This can be easily done following [35]. We first notice that the center of mass position x_m of the multi-instanton solution can always be defined as the trace part of the a_m matrix, as in section 3, see (3.9). From (5.3) one can then easily see that x_m parameterize a point in the $\mathbb{R}^4/\mathbb{Z}_p$ space if we test the geometry with regular D-instantons. If instead we use fractional D-instanton probes, x_m is simply frozen to zero.

In order to extend the ansatz (2.5) to the ALE situation we should first give a covariant definition for b in (2.2)

$$
b = \sigma^n_{[2]\times[2]}\mathbf{1}_{[k]\times[k]}\nabla_n\Delta\,, \qquad \bar{b} = \bar\sigma^n_{[2]\times[2]}\mathbf{1}_{[k]\times[k]}\nabla_n\bar\Delta\,, \tag{5.5}
$$

where $\nabla_n = \partial_n + A^{\mathcal{T}}_n$ is the covariant derivative respect to a connection $A^{\mathcal{T}}_n$ in the tautological bundle $\mathcal{T}$. The ADHM ansatz for the gauge potential (2.5) and adjoint fermionic zero modes are modified now to [35, 36]

$$
\begin{aligned}
A_n &= U^\dagger \nabla_n \mathrm{U}\,,\\
\Psi^A &= \bar{U}\left(\mathcal{M}^A f \bar\sigma^n \nabla_n \bar\Delta - \nabla_n \Delta \sigma^n f \bar{\mathcal{M}}^A\right)\mathrm{U}
\end{aligned}
\tag{5.6}
$$

which covariantly generalize (2.5). The matrices U and $\bar{U}$ are defined as before as the kernels of $\Delta, \bar\Delta$. Moreover, the ADHM matrices decompose under Γ as a collection of maps

$$
\Delta^q_{[N_q+2k_q]\times[2\hat{k}_q]} : W_q + (V_q \otimes \mathbf{1}_{2\times2}) \longrightarrow Q \times V_q\,. \tag{5.7}
$$

These properties can be used to relate the Chern character of the instanton bundle to the Chern characters of the individual bundles $\mathcal{T}_q$. The instanton bundle is specified then by giving the first and second Chern class

$$c_1 = \sum_q \left(N_q + 2k_q - 2\hat{k}_q\right) c_1\left(\mathcal{T}_q\right),$$
$$c_2 = \sum_q \left(N_q + 2k_q - 2\hat{k}_q\right) c_2\left(\mathcal{T}_q\right) + \frac{K}{|\Gamma|}. \tag{5.8}$$

In particular an instanton solution with vanishing first Chern class is given by

$$N_q + 2k_q - 2\hat{k}_q = 0, \qquad \text{for } q > 0. \tag{5.9}$$

This is a highly non-trivial constraint on the allowed values of (k_q, N_q). Notice that only in this case the instanton number defined as $K/|\Gamma| = K/p$ coincides with the second Chern class. In [35] the reader will be able to find detailed discussions of various cases. Here we simply recall the results in the simplest context: the SU(2) gauge bundle on the Eguchi-Hanson blown down space $\mathbb{R}^4/\mathbb{Z}_2$. Solutions to (5.9) in this case are given by either $\vec{N} = (2,0)$, $\vec{k} = (k,k)$ or $\vec{N} = (0,2)$, $\vec{k} = (k-1,k)$. They lead to instanton solutions with integer and half-integer second Chern class, respectively, as can be easily see from (5.8). The dimension of the multi-instanton moduli space can be read from (5.4) and turns out to be, respectively equal to $8k$ and $8k-4$, in agreement with [37]. A computation of the partition function for the lowest value of the Chern class, $c_2 = 1/2$, was carried out in [37] yielding the bulk contribution to the Euler number of the moduli space.[15]

Having described the main topological properties of the instanton solutions described by (5.3), we come back to the study of the moduli space geometry. The low energy effective action (2.10) written in terms of (5.3) gives again a very tractable description of the physics around the multi-instanton background. Since the whole SO(6) $\mathcal{R}$-symmetry is clearly preserved by the projection, this action describes instantons in an $\mathcal{N} = 4$ gauge theory, living on an ALE space.

It is a straightforward exercise to apply the techniques of the previous section to show that for a regular instanton probe the center of mass degrees of freedom factor out from the multi-instanton measure and describe a point in an euclidean $S^5 \times AdS_5^E/\mathbb{Z}_p$ space, with $\mathbb{Z}_p$ acting on the four-dimensional $\mathbb{R}^4$ boundary of the AdS_5^E. In the case of fractional instantons only the origin of this boundary space is clearly probed. The crucial observation is that, unlike in the non-conformal situations considered in the previous section, no r dependence comes now from the determinants $(\det_{2k_q} W \det_{4k_q} \chi) \sim y^{4k_q}$ in (3.18), since the orbifold projection acts in the same way on the W and χ fields, balancing their contribution against each other. On the other

[15] It is known and rigorously proven that for $c_2 = 1/2$ the moduli space is a copy of the base manifold (the Eguchi-Hanson manifold) whose bulk contribution is 3/2.

hand also the various contributions to the measure coming from the Jacobian of the scalings (3.9) cancel, leaving a conformal $AdS_5^E/\mathbb{Z}_p \times S_5$ geometry.

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A. On the spinorial representations of SO(10)

In section 2 we have seen how, by modding the internal symmetry group, the number of supersymmetric charges is reduced and we have discussed the case of a $\mathcal{N} = 4$ theory going to a $\mathcal{N} = 2$. In section 4 this latter theory has been twisted allowing a topological interpretation. The following material will help the reader to convince himself that in this reduction the right quantum numbers for the $\mathcal{N} = 2$ theory are generated.

To avoid the introduction of more formalism, we will use the notations of [38], to which we refer the reader for a complete treatment of the subject. As it is well known, the highest weight of a representation, $\mu = \sum q^i \mu^i$, can be built as a tensor product starting from the fundamental weights μ^i. The representations whose weights are the μ^i are the fundamental representations D^i. For SO(10) we concentrate on two such representations, D^4 and D^5 with weights $\mu^4 = (e^1 + \cdots - e^5)/2$ and $\mu^5 = (e^1 + \cdots + e^5)/2$, where the $\pm e^i \pm e^j, i \neq j$ are the roots of the algebra. Both representations are 16 dimensional and if we denote by $1/2 \sum_i \eta_i e^i$ a generic weight (with $\eta^i = \pm 1$), we have that for D^4, $\prod_i \eta_i = -1$, while for D^5, $\prod_i \eta_i = 1$. This means that D^4 (D^5) has negative (positive) parity, since the η_i can be chosen to be the eigenvalues of some σ_3^i Pauli matrices out of which the Dirac Γ_{11} matrix (in the Clifford representation) can be built as the tensor product $\Gamma_{11} = \otimes_{i=1}^5 \sigma_3^i$. Let us now see what happens to D^4, D^5 under the regular maximal subgroup, i.e. the subgroup whose rank is the same as that of the original group. These subgroups are obtained by studying the extended Π system of the algebra. In our case we obtain $\mathrm{SO}(10) \to \mathrm{SO}(6) \times \mathrm{SO}(4) \cong \mathrm{SU}(4) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The simple roots of SO(6) are now

$$\begin{aligned}
\alpha^1 &= e^1 - e^2 \, , \\
\alpha^2 &= e^2 - e^3 \, , \\
\alpha^0 &= -e^1 - e^2
\end{aligned} \tag{A.1}$$

and those of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ are

$$\begin{aligned} \alpha^4 &= e^4 - e^5 \,, \\ \alpha^5 &= e^4 + e^5 \,. \end{aligned} \tag{A.2}$$

The weights of D^5 are now divided into two sets $\{\eta_1\eta_2\eta_3 = 1, \eta_4\eta_5 = 1\}, \{\eta_1\eta_2\eta_3 = -1, \eta_4\eta_5 = -1\}$. Recapitulating, we have seen that $D^5 \to (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$. Furthermore, given that the maximal regular subgroup of $\mathrm{SU}(4) \sim \mathrm{SO}(6)$ is $\mathrm{SO}(4) \times \mathrm{SO}(2) \sim \mathrm{SU}(2) \times \mathrm{SU}(2)' \times \mathrm{U}(1)$ it follows that the U(1) coming from the decomposition of the $\mathbf{4}$ with positive parity $\eta_1\eta_2 = +1$ has U(1) quantum number $\eta_3 = +1$.

The positive parity Weyl-Maiorana spinor in ten space-time dimensions is

$$\psi = \sqrt{\frac{\pi}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \mathcal{M}^A_\beta \\ 0 \end{pmatrix} + \sqrt{\frac{\pi}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \lambda^A_{\dot\beta} \end{pmatrix} \tag{A.3}$$

$\mathcal{M}^A_\beta, \lambda^A_{\dot\beta}$ are the $(\mathbf{4},\mathbf{2},\mathbf{1})$ and $(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$ parts, respectively.

Let us now discuss the modding: we decide to label by $\mu = 7,\ldots,10$ the directions longitudinal to our D3-brane. The Γ subgroup is going to act on the transverse $\hat a = 1,2,4,5$ directions. We also consider $\Gamma = \mathbb{Z}_2$ to recover pure $\mathcal{N} = 2$ SYM_4. In this case the action of the $\mathbb{Z}_2$ is that of an inversion, A, which transforms the spinor (A.3) as $\psi'(x') = \psi'(A\cdot x) = S(A)\psi(x)$. Requiring the invariance of the Dirac's equation we find $S^{-1}\Gamma^{\hat a}S = -\Gamma^{\hat a}$ with $\hat a = 1,2,4,5$. As it is well known, the solution to this equation is given by the parity matrix.

A convenient representation of the ten-dimensional Clifford algebra is given by

$$\Gamma_a = \begin{pmatrix} 0 & \Sigma^a_{AB} \\ \bar\Sigma^a_{AB} & 0 \end{pmatrix} \otimes \gamma^5 \,, \qquad \Gamma_\mu = 1_{8\times 8} \otimes \gamma^\mu \,, \tag{A.4}$$

where the γ^μ's define a four-dimensional Clifford algebra in the longitudinal directions

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.5}$$

By using (A.4) it is now easy to see that the condition of parity invariance in the $\hat a = 1,2,4,5$ directions boils down to setting to zero those fermionic components with $A = \bar A = 3,4$. For what the vector is concerned, $\chi' = A \cdot \chi = -\chi$ for the components $\mu = 1,2,4,5$.

Implementing these considerations in (2.10) one easily obtains the measure for the ADHM construction with $\mathcal{N} = 2$. Modding out by a discrete subgroup turns out to be a very effective way to deduce lagrangians with a lower number of supersymmetries.

B. Relationship between $\mathcal{N} = 2$ and $\mathcal{N} = 4$ topological theories

In section 3 we showed how to obtain a $\mathcal{N} = 2$ supersymmetric theory out of a $\mathcal{N} = 4$ one by imposing Γ invariance. The same exercise can be carried on with topological theories. Our starting point will be the $\mathcal{N}=4$ topological theory studied in [21], which we rewrite here as

$$S = Q\,\mathrm{tr}_k \left(\frac{1}{4}\eta[\phi,\bar\phi] + \vec{H}\cdot\vec{\chi} - i\vec{\mathcal{E}}\cdot\vec{\chi} - \frac{1}{2}\sum_{l=1}^{6}\left(\Psi_l^\dagger\bar\phi\cdot B_l + \Psi_l\bar\phi\cdot B_l^\dagger\right)\right), \qquad \text{(B.1)}$$

where $\vec{H} = (H^{(a)}_{\dot\alpha}, H^{(f)}_{\dot\alpha}, iD^c)$ are the auxiliary fields which implement the constraints $\vec{\mathcal{E}} = (\mathcal{E}^{(a)}_{\dot\alpha}, \mathcal{E}^{(f)}_{\dot\alpha}, \mathcal{E}^c)$ and $\vec{\chi} = (\chi^{(a)}_{\dot\alpha}, \chi^{(f)}_{\dot\alpha}, i\lambda^c)$ their fermionic superpartners, while $B_l = (\phi^s, w_{\dot\alpha}, a_m)$ and Ψ_l are the corresponding fermionic superpartners. The quiver projection of section 3.2 implies that the ϕ^s fields are set to zero; then, also their fermionic superpartners and the associated auxiliary fields are dropped, and we are left with

$$S = Q\,\mathrm{tr}_k \left(\frac{1}{4}\eta\left[\bar\phi,\phi\right] - \lambda_c D^c + \lambda_c\mathcal{E}^c - \mathcal{M}_m\left[\bar\phi, a_m\right] + \frac{1}{2}\bar\mu^{\dot\alpha}\bar\phi w_{\dot\alpha} + \frac{1}{2}\mu^{\dot\alpha}\bar\phi\bar{w}_{\dot\alpha}\right), \quad \text{(B.2)}$$

where $\mathcal{E}^c$ can be identified with the twisted bosonic constraint, last equation in (4.2). Finally, by adding a vev to the scalar fields and making the simple rescalings $\phi\to 2\phi$, $\eta\to\eta/g_0^2$, $D^c\to D^c/g_0^2$, we get the action (4.4).

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